# *An Introduction to Lazy Cops and Robbers on Graphs*

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 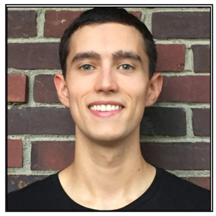 

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There's a robber on the loose! As the police chief, you must give directions to your squadron of cops. The goal is to describe a winning strategy that allows them to catch the robber no matter what he tries to do to escape. How many cops do you need in your squadron to achieve such a strategy? And what if your cops are "lazy," meaning that only one may move each turn, instead of all of them at once? In this article, we describe a recently proposed variant of the classic Cops and Robbers game on graphs known as Lazy Cops and Robbers [**5, 6, 23**]. Specifically, we explore how the lazy variant affects the number of cops needed to win. We present examples where the ordinary and lazy game play out similarly, as well as examples where they play out strikingly differently. We also pose several open problems for investigation.

## Cops and Robbers on graphs

Cops and Robbers is a pursuit–evasion game played on graphs. The standard form was introduced independently by Quilliot in 1978 [**25**] and by Nowakowski and Winkler in 1983 [**23**]. The rules are given below. Throughout, we consider only finite, simple,

http://dx.doi.org/10.4169/college.math.j.48.5.322
MSC: 05C57, 91A43

 

connected, undirected graphs. (For graph theoretic terminology that is not defined here, see any text on graph theory, e.g., [**29**].)

- Start with a graph, one robber, and $k$ cops for some positive integer $k$.
- The cops each choose a beginning vertex (more than one on a vertex is allowed).
- Then, the robber chooses a beginning vertex.
- In a move, each cop may move along an edge to an adjacent vertex or stay put.
- Then, the robber may move along an edge to an adjacent vertex or stay put.
- The teams continue like this, alternating turns.
- Cop victory condition: If at any time during gameplay, a cop occupies the same vertex as the robber, then the cops win.
- Robber victory condition: If there exists a strategy for the robber whereby he can move around the graph and avoid the cops forever, then the robber wins.

Given a graph, how many cops are needed to guarantee that they catch the robber? A trivial answer comes from having the cops occupy every vertex. But perhaps fewer cops can win by playing more judiciously. Indeed, there must be some minimum value $k$ such that a set of $k$ cops can be told how to play such that they are guaranteed to win after a finite number of moves, no matter what the robber does. This is known as a winning strategy for the cops.

Given a graph $G$, let $c(G)$ denote the *cop number* of $G$, the minimum number of cops required to guarantee the existence of a winning strategy for the cops. In general, to prove $c(G) = k$, two arguments are needed: One must show that (a) $k$ cops have a winning strategy, but (b) against $k - 1$ cops, the robber has a winning strategy.

Here are some examples of cop numbers for graphs. Figure 1(a) shows $P_4$, the path graph of length four (the number of edges). For any path, $c(P_n) = 1$ because a single cop can start at one end and move toward the other end. At some point, he must catch the robber.

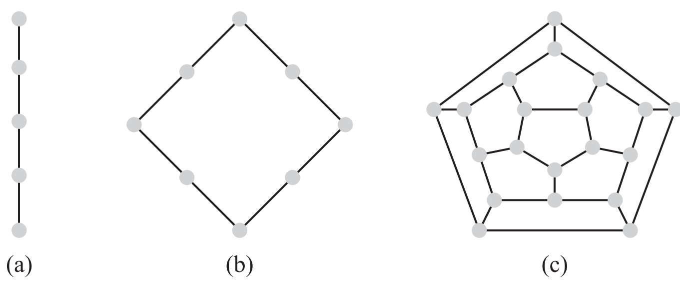


**Figure 1.** Examples of graphs for which the cop number is 1, 2, and 3, respectively: (a) the path $P_4$, (b) the cycle $C_8$, and (c) the dodecahedron.

Figure 1(b) shows $C_8$, the cycle graph of length eight. As long as $n \geq 4$, we find $c(C_n) = 2$. One cop is insufficient since the robber may stand still until the cop is adjacent and then respond by moving one step farther away. That is, the robber can ensure the distance between him and the cop is always at least two. On the other hand, two cops suffice since they can start on one vertex and travel around the cycle in opposite directions, squeezing the robber between them.

Figure 1(c) shows the dodecahedron graph, a planar representation of the vertices and edges of the Platonic solid of the same name. Theorems 1 and 2 below imply that the cop number of this graph is three; we encourage the reader to discover a strategy whereby three cops can catch the robber. And as a preview of a later topic, we encourage you to find a winning strategy for three cops in which only one cop moves on each turn.

Aigner and Fromme proved several significant results in 1984 [**1**]. For one, they showed that there exist graphs with arbitrarily large cop numbers. Specifically, they proved the following. Recall that the minimum degree among the vertices of a graph $G$ is denoted $\delta(G)$.

**Theorem 1.** *If a graph G has no 3-cycles and no 4-cycles, then* $c(G) \geq \delta(G)$.

They followed this by inductively constructing $k$-regular graphs with no 3-cycles and no 4-cycles, for any integer $k$. These results together show that graphs exist with arbitrarily large cop number.

Another result of theirs showed, in contrast, that one may impose conditions on a graph that bound the cop number from above.

**Theorem 2.** *If G is a planar graph, then* $c(G) \leq 3$.

This is truly remarkable! How does a condition about not crossing edges relate to the cop number? First, Aigner & Fromme showed that a cop may "patrol" a path inside the graph, in the sense that the cop can move along that path to prevent the robber from ever stepping onto it. If one designates two cops to patrol two paths that form a cycle, then the assumption that $G$ is planar means that those patrolling cops divide the graph into an inside and outside. The robber must be in one or the other, and by choosing their moves carefully, three cops can keep shrinking the territory in which the robber is roaming, eventually trapping him.

Cops and Robbers remains an active area of research. The most significant open problem, known as Meyniel's conjecture [**4**], claims that $c(G) = O(\sqrt{n})$; that is, asymptotically speaking, the cop number of a graph is bounded above by some constant multiple of the square root of the number of vertices in the graph.

**Variations.** Researchers have also investigated some variations on the standard game. For instance, what if we remove players' knowledge of the others' location and require one or both sides to move "in the dark" [**15**]? What if we allow the robber to move along several edges at a time on his turn [**2**]? What if the cops must stay close to each other [**12**]? What if the cops can capture the robber from a greater distance [**10**]? And when the cops are guaranteed to win, can we determine the "capture time," the number of moves it will take [**17**]? We highly recommend the monograph by Bonato and Nowakowski [**11**] for much more on the standard game and some variants.

In a recent article, Offner and Ojakian explored variants that only allow a certain proportion of the cops to be "active" during their turn [**24**]. The extreme case, where only one cop may move and the rest must stay put, has since gained the name Lazy Cops and Robbers [**5, 6**]. The rules differ from those of the standard game only on the cops' turn, in which one and only one cop can be chosen to move. This leads one to define and investigate the lazy cop number of a graph, denoted $c_L(G)$, which is the minimum number of cops required to guarantee the existence of a winning strategy for lazy cops.

Publications have investigated the asymptotics of the lazy cop numbers of hypercubes and random graphs [**5, 6**] as well as the complexity class of determining lazy cop numbers [**16**], but these papers are rather technical. This led us to investigate the game of Lazy Cops and Robbers from a more basic standpoint and, subsequently, to share some of our findings here. Everything discussed in this article was developed during an undergraduate summer research project by the authors (two students and a faculty mentor). We believe this topic can be investigated in a course on graph theory or an introduction to formal proofs, showing that undergraduate students can dive in and conduct novel mathematical research.

Before reading on, we encourage the reader to return to the examples in Figure 1 and determine their lazy cop numbers. What is the lazy cop number of an arbitrary path or cycle? Is it obvious whether $c_L = c$ or $c_L > c$? Remember that we are interested in how many cops are needed to win, not how many moves it will take.

## Comparing ordinary and lazy cops

For an arbitrary graph $G$,

$$c(G) \leq c_L(G) \leq \gamma(G) \tag{1}$$

where $\gamma(G)$ denotes the domination number of $G$, the minimum size of a dominating set $S$—a set of vertices such that every vertex in $G$ is either in $S$ or adjacent to a vertex in $S$. The left-hand inequality says that, if the cops are lazy, then we certainly need at least as many as $c(G)$ of them to win (and possibly more). Alternatively, one can argue that a squadron of ordinary cops may always choose to play lazily. The right-hand inequality states that an obvious way to win is to position the cops so that they guard every vertex. By positioning the cops on the vertices of a dominating set $S$, they can guarantee victory in one move, which is achievable even if they are lazy.

Notice that if $c(G) = 1$, it must also be that $c_L(G) = 1$; with only one cop, there's no notion of laziness. It follows that if $c_L(G) = 2$, then $c(G) = 2$. This observation will be relevant in proving Theorem 3.

Next, we will see two examples where $c = c_L$, showing that the left-hand inequality of (1) may be satisfied at equality. Then, we will see two examples where $c < c_L$, including one where $c_L = \gamma$, showing that the right-hand inequality of (1) may also be satisfied at equality.

## Lazy cops may be equally effective

For the next example, the graph $G$ will be a lattice grid wrapped around the surface of a cylinder, as if it were printed on the label of a can. In graph theoretic terms, $G$ is the Cartesian product of a cycle with a path, denoted $C_m \square P_n$. (The notation $\square$ is chosen since the Cartesian product of an edge with an edge is precisely a square; see Neufeld and Nowakowski [**22**] for more about graph products and Cops and Robbers.) The cycle factor indicates that one may move around the label of the can, and the path factor indicates one may move up and down the label but the top and bottom are not connected. To avoid trivialities, we will assume that the cycle factor has $m \geq 4$ vertices and the path factor has $n + 1 \geq 2$ vertices. (Recall that the path $P_n$ has $n$ edges and $n + 1$ vertices.)

We refer to levels $L_1, \ldots, L_{n+1}$ of the graph, where $L_i$ denotes all the vertices on the same cycle factor corresponding to vertex $i$ along the path factor. Think of these as the rings around the can, with $L_1$ at the bottom and $L_{n+1}$ at the top. Also, we refer to the robber's shadow $S_i$, the vertex on $L_i$ that is in the same path factor as the robber. (In the proof below, the robber will always be at or above his shadow $S_i$, in the sense that the vertex the robber actually occupies is on $L_j$ for some $j \geq i$.) And we will also have occasion to refer similarly to the shadow of a particular cop on level $L_i$; this is denoted by $S_i^\star$ (the star metaphorically indicating a cop).

**Theorem 3.** *For $G$ a cylindrical lattice, $c_L(G) = c(G) = 2$.*

*Proof.* First, observe that one cop cannot win because the robber can stay on $L_1$ and always escape. No matter where the cop is in $G$, the robber will just play against the cop's shadow $S_1^\star$. Since this amounts to the robber playing against one cop on a cycle (with $m \geq 4$ vertices), the robber has a winning strategy. Thus, $c_L(G) \geq 2$.

Next, we provide a winning strategy for two lazy cops. Name them the good cop $C^\star$ and the bad cop $C$. They will inductively advance up the levels in what we call stages. We define stage $i$ to mean

- both cops occupy vertices on $L_i$,
- the robber $R$ occupies some vertex on $L_j$ for some $j \geq i$,
- the good cop $C^\star$ occupies the robber's shadow $S_i$, and
- it is the robber's turn.

To start, place the cops anywhere on $L_1$. The robber then places himself anywhere in the graph. The cops ignore the actual location of the robber and play against his shadow $S_1$. Since $c_L(C_m) = 2$, one of the cops must eventually capture $S_1$. Call that cop the good cop $C^\star$; now the cops have entered stage 1, the base case of our induction. (Note: If the robber actually occupies $S_1$, then the cops have already won, so we may assume the robber is on $L_2$ or higher.)

For the induction hypothesis, suppose that the cops have entered stage $i$ for some $1 \leq i \leq n$. By assumption, it is the robber $R$'s turn. By considering cases based on $R$'s moves, we will show how the cops can advance to stage $i + 1$. With the observation that stage $n + 1$ is a cop victory, as the robber cannot be any higher than $L_{n+1}$, this will complete the proof.

The cases are based on whether $R$'s move changes his shadow $S_i$. If it does not, this means he stayed put or moved vertically (up or down the grid). If this puts him one step away from $C^\star$, then the cops can win, so they should certainly take that move. So, in what follows, we will assume that the game has not yet ended due to the robber's dumbness; we will show that he is still eventually forced to make a losing move.

(1) Suppose $R$'s first move does not change his shadow $S_i$. Send the bad cop $C$ up to $L_{i+1}$ by moving in his path factor. Or, if $R$ is now on $L_{i+1}$, let $C^\star$ capture him instead. This way, in case (1b), we may assume $R$ is on $L_j$ for $j \geq i + 2$.

(1a) Suppose $R$ responds by again not changing his shadow. Similarly, send the good cop $C^\star$ up to $L_{i+1}$. Now, the cops are in stage $i + 1$. (Or $R$ is captured, but see the note above.)

(1b) Suppose $R$ responds by changing his shadow (i.e., moving horizontally). Send $C^\star$ up to $L_{i+1}$. The cops are not quite in stage $i + 1$ since $C^\star$ is no longer on $R$'s relevant shadow $S_{i+1}$. Now, as the robber moves, the two cops play against $S_{i+1}$ but in the particular following way.

If $R$ ever moves horizontally "backward" and puts $S_{i+1}$ on $C^\star$, then the cops will pass their turn and enter stage $i + 1$. Or, if $R$ ever stays put or moves vertically, then his shadow $S_{i+1}$ is just one step away from $C^\star$; the good cop will move over to capture $S_{i+1}$, and the cops enter stage $i + 1$.

The only way $R$ can avoid those two scenarios is by continually moving his shadow away from $C^\star$. In response, $C^\star$ moves toward $S_{i+1}$ to remain one step away. Eventually, the only way $R$ can avoid one of the two scenarios mentioned above is to put $S_{i+1}$ on $C$, in which case the cops pass their turn, and the good and bad cops swap roles upon entering stage $i + 1$. See Figure 2 for a diagram of this scenario.

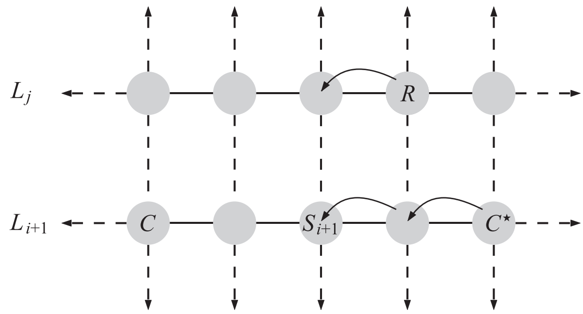


**Figure 2.** Case (1b): The cops have moved up to level $L_{i+1}$ but need to capture $R$'s shadow $S_{i+1}$.

(2) Suppose $R$'s first move is horizontal, so his shadow $S_i$ changes. Again, we choose to send $C$ up to $L_{i+1}$.

(2a) Suppose $R$ responds by not changing his shadow (i.e., moves vertically). If $R$ is currently on $L_j$ for $j \geq i+2$, then the cops can afford to send $C^\star$ up to $L_{i+1}$. This yields a situation analogous to case (1b), shown in Figure 2, where the cops play against and eventually capture $S_{i+1}$.

Otherwise, if $R$ is currently on $L_{i+1}$, then the cops cannot send $C^\star$ to $L_{i+1}$ because $R$ could respond by "passing through" the cops' defense to $L_i$, thereby ruining our induction. Instead, the cops will send the bad cop $C$ toward $R$ along $L_{i+1}$ in the following way. On level $L_{i+1}$ (which is a cycle factor), identify $C$, $R$, and the shadow $S^\star_{i+1}$ of $C^\star$. Then, identify the path from $C$ to $S^\star_{i+1}$ that contains $R$. Move $C$ to shorten that path. That is, $C$ moves horizontally in the direction of $C^\star$ that keeps $R$ between them. If the robber ever passes his turn, then $C$ moves toward him in this manner, ensuring that he cannot simply pass indefinitely: The robber must eventually make a move, and the next paragraph shows how to respond.

If $R$ moves back onto $S^\star_{i+1}$, then $C^\star$ may move up to capture him. If $R$ moves up to $L_{i+2}$, then $C^\star$ can afford to move up to $L_{i+1}$, and the cops are again in a situation analogous to case (1b). And if $R$ moves horizontally away from $S^\star$, then $C^\star$ mimics his move, stepping along $L_i$ in the same direction as $R$, preventing $R$ from moving down and "passing through" the cops. Since $C$ is on $L_{i+1}$, the robber $R$ cannot afford to make such a move indefinitely without bumping into $C$ and losing. Thus, after a finite number of moves, the cops have advanced to stage $i+1$ (or have won).

(2b) Suppose $R$ responds by again moving his shadow. If $R$ is currently on $L_{i+1}$, then the cops are in a situation analogous to the one described in the final paragraph of case (2a): $C^\star$ responds by moving in the same direction as $R$ until $R$ eventually bumps into $C$ or decides to retreat up to $L_{i+2}$, allowing $C^\star$ to move up to $L_{i+1}$ (and then they are in case (1b) again). (Or, if the cops were in stage $n$, then the robber has no escape since there is no $L_{n+2}$.)

Otherwise, if $R$ is currently on $L_j$ for $j \geq i+2$, then the cops respond by sending $C^\star$ in the same direction. This again puts them in a situation analogous to the one described in the final paragraph of case (2a). The only difference here is that $R$ may not actually bump into $C$; instead, $R$'s shadow $S_{i+1}$ will. In that case, $C^\star$ will move up to $L_{i+1}$, and then the cops swap roles upon entering stage $i+1$.

This completes the induction step. By following this strategy, the cops may advance up the cylinder and trap the robber at the top level (if not earlier).

Finally, as we observed from (1), $c_L(G) = 2$ implies $c(G) = 2$. ∎

This proof indirectly shows that $c(G) = 2$. We encourage the reader to try to find a winning strategy for two ordinary cops. Does it differ significantly from the lazy cops' strategy? Do you think it shortens the capture time, i.e., the number of moves required to win?

For another example, Aigner & Fromme's Theorem 1 above shows that the Petersen graph has $c \geq 3$ since the graph is 3-regular and contains no 3- or 4-cycles. Moreover, observe that the vertices occupied by the three cops in Figure 3 form a dominating set. By inequality (1), this means $c = c_L = \gamma = 3$.

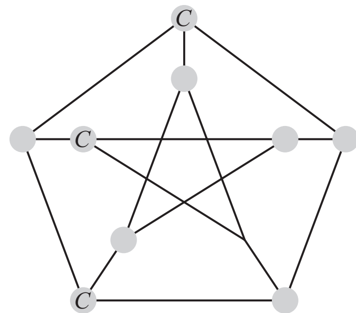


**Figure 3.** The Petersen graph has $c = c_L = \gamma = 3$.

Recently, Beveridge et al. proved that the Petersen graph is the unique smallest graph with $c = 3$; that is, among all graphs with $c = 3$, the Petersen graph has the fewest number of vertices, and all other graphs on 10 vertices have $c \leq 2$ [**8**]. Inspired by that result, we worked to discover an analogous result for lazy cops; in fact, we have proven that the smallest graph with $c_L = 3$ is the $3 \times 3$ rook's graph, $R_3$ [**26**]. (Skip to the next section for a definition.) Since $R_3$ has 9 vertices while the Petersen graph has 10, this also means that $R_3$ is the unique smallest graph for which $c_L > c$.

Refer back to Figure 1(c) for the dodecahedron graph. Aigner & Fromme's Theorems 1 and 2 above indirectly show that $c = 3$ for this graph because it is 3-regular with no 3- or 4-cycles and it is planar. Interestingly, $c_L = 3$ for this graph, as well; this can be shown by placing three cops on the graph, determining where the robber could safely start, and showing that, in all possible cases, the cops can catch him after just a couple of moves, even with only one cop moving at a time. (Careful initial placement of the cops can reduce the number of cases.) However, the domination number $\gamma = 6$ for this graph, so unlike the Petersen graph, this is an example where $c = c_L < \gamma$.

## Lazy cops may be strictly less effective: One threat at a time

In contrast to the examples in the preceding section, we have found several examples of graphs where $c_L > c$, including some where the difference is large. We will show two such examples here. Coincidentally, both are based on the game of chess: The underlying graphs represent the legal moves of a particular chess piece. One reason we thought to investigate these graphs is that the mechanics of the Lazy Cops and Robbers variant are similar to chess; although the cops have an entire squadron of pieces at their disposal, they may only move one piece at a time.

Let $R_n$ denote the rook's graph on an $n \times n$ board, where the vertices represent the squares of the board and the edges correspond to legal moves by a rook. Recall that a rook may move any number of spaces either horizontally or vertically, so vertices are adjacent if and only if they lie in the same row or column. For these graphs, the next two results show that ordinary and lazy cop numbers are as far apart as they could possibly be!

**Theorem 4.** *For any* $n \geq 2$, $c(R_n) = 2$.

**Theorem 5.** *For any* $n \geq 2$, $c_L(R_n) = \gamma(R_n) = n$.

The proofs of these results are straightforward, so we leave them as instructive exercises for the reader, but we will offer a few helpful comments. Proving Theorem 4 amounts to exhibiting a winning strategy for two cops, and this can be achieved in only one move! A rook has only two directions of motion, so the cops merely need to "capture" both directions in one fell swoop.

Proving Theorem 5 involves two steps: (1) $n$ cops can dominate the graph (this is trivial), and (2) the robber has a winning strategy against $n - 1$ cops. This second step uses both the pigeonhole principle and the recognition that lazy cops may only make one threat at a time. That is, the robber will never find himself adjacent to two cops on his turn. Why? If that were the case, that would mean one cop was already adjacent to the robber after his turn and then a second moved to be adjacent, as well. But instead of moving the second cop closer, on the cops' turn, they would have moved the first cop to capture the robber instead.

This important observation will be used in our next example. Indeed, we find this to be a significant difference between the ordinary and lazy games: Ordinary cops may make several threats at once, whereas lazy cops are restricted to one threat at a time.

The previous two theorems show that the ordinary and lazy versions of the game play out quite differently on rook's graphs. Now we will prove that a similar phenomenon occurs for $n \times n$ queen's graphs, $Q_n$. Recall that a queen may move any number of spaces either horizontally, vertically, or diagonally. We will not prove exact values but show that $c$ is bounded by a constant while $c_L$ grows asymptotically linearly with $n$. The notation $\lceil \cdot \rceil$ denotes the integer ceiling.

**Theorem 6.** *For any* $n \geq 7$, $c(Q_n) \leq 4$.

**Theorem 7.** *For any* $n \geq 7$, $\left\lceil \frac{n}{3} \right\rceil \leq c_L(Q_n)$.

(As we will see later, we know exact values for $n \leq 6$, thus the $n \geq 7$ condition.)

*Proof of Theorem 6.* Initially, place one cop at each of the four corners. The robber starts anywhere. On the cops' first turn, send one to the robber's row, column, and two main diagonals by identifying where those lines intersect the border of the grid and choosing the cops' moves accordingly. (See Figure 4 for an example.) Regardless of the robber's next move, he will be caught. ■

For small $n$, one or two or three ordinary cops suffice, but we have found that four cops are necessary for $n \geq 19$. We do not necessarily think that $n = 19$ is the smallest queen's graph such that $c = 4$, but it is the best cutoff we have found so far. (See the concluding section for more information.)

*Proof of Theorem 7.* We exhibit a winning strategy for the robber against $k$ lazy cops on $Q_n$ (with $k$ to be determined in the proof), using the observation about one threat at a time. Suppose it is the robber's turn. If he is not threatened by a cop, then he may safely pass his turn, so have him do so. Otherwise, there is exactly one threatening cop. We will determine how many other cops would be needed to block some of the robber's possible moves.

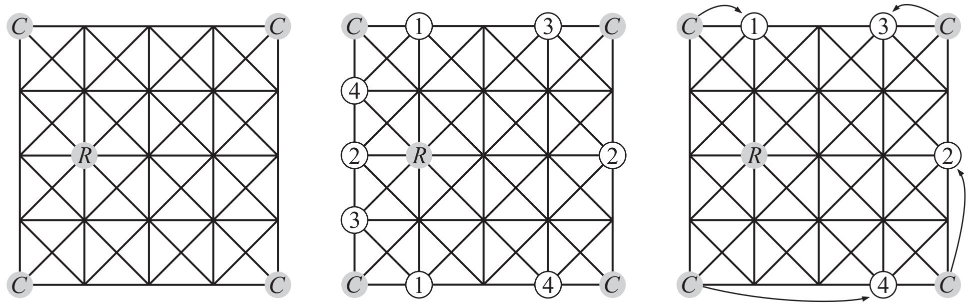

**Figure 4.** Four cops have a winning strategy on the queen's graph $Q_n$. (a) Initial placement of the cops on the corners. (b) Where $R$'s four lines of movement hit the border. (c) The cops can capture all four lines in one turn.

Regardless of the location of the robber and the threatening cop, either the robber's row or column is unoccupied by a cop (or possibly both, if the threat is diagonal). Choose one such line and call it $L$; the robber will look to escape somewhere along $L$. Of the $n$ squares on $L$, how many are safe and how many are guarded by cops?

The robber's current square is under threat by a cop. That same cop can guard at most two other squares on $L$: Draw four lines through that cop, horizontal, vertical, and two main diagonals. One of those lines will be parallel to $L$, while the other three may intersect $L$. (If the robber and cop are near the edge of the board or if they are on differently colored squares, then the diagonal(s) may not intersect.) One of those possible intersections is already accounted for: the robber's current square. Overall, then, there are at least $n - 3$ squares on $L$ still available for his escape.

Conveniently, the argument above about guarding at most three squares on $L$ also applies to any cop! This is because no other cop could be on $L$ since this would indicate two simultaneous threats. Thus, by drawing the same four lines through each cop and noting that one is parallel to $L$ while the other three may intersect $L$, we have the following fact. A cop not on $L$ can guard at most three squares on $L$. (Indeed, this fact is true for any line on the board; there is nothing special, in general, about this line passing through the robber.)

We had at least $n - 3$ squares available on $L$ for the robber's escape, and any other cop can guard at most three of them. Notice that $\lceil \frac{n-3}{3} \rceil - 1$ is the largest integer strictly less than $\frac{n-3}{3}$ (thus the ceiling function instead of the floor). With exactly this many cops, they cannot even guard all vertices on $L$, let alone any others to which the robber might escape. Accounting for the one threatening cop, as well, we find that

$$c_L(Q_n) > 1 + \left( \left\lceil \frac{n-3}{3} \right\rceil - 1 \right) = \left\lceil \frac{n-3}{3} \right\rceil.$$

Since this is a strict inequality (i.e., the cops lose), then this implies that

$$c_L(Q_n) \geq \left\lceil \frac{n-3}{3} \right\rceil + 1 = \left\lceil \frac{n}{3} \right\rceil. \qquad \blacksquare$$

Notice that this proof considers how many cops are needed to block some particular subset of the robber's possible moves. We believe that a more careful consideration of the possible placements of the nonthreatening cops may yield a larger lower bound.

Meanwhile, we also know that $c_L \leq \gamma$. The study of combinatorial problems on the chessboard has been an active area of research for decades, but there are still some open problems, including an exact formula for $\gamma(Q_n)$. At the time of this article, the best known bounds are

$$\frac{n-1}{2} \le \gamma(Q_n) \le \left\lceil \frac{2n}{3} \right\rceil.$$

The lower bound is due to Spencer and the upper bound to Welch [**28**]. Combining our result above with these bounds shows that

$$\left\lceil \frac{n}{3} \right\rceil \le c_L(Q_n) \le \left\lceil \frac{2n}{3} \right\rceil. \tag{2}$$

However, it is known that Spencer's bound is satisfied at equality for certain values of $n$, and it is also conjectured that the true values of $\gamma$ lie closer to Spencer's bound, in general, than Welch's. Thus, there are several reasons to believe that the bounds in (2) are not as sharp as they could be. At least, we have shown that the lazy cop number of the queen's graph grows with the board's size (more specifically, with the square root of the number of vertices), whereas the ordinary cop number cannot exceed four.

## Related results and conjectures

We encourage the reader to investigate bishop's graphs $B_n$, which are essentially modified rook's graphs. One can show that $c(B_n) = 2$ and $c_L(B_n) = \gamma(B_n) = \lfloor \frac{n+1}{2} \rfloor$, another instance where $c = 2$ yet $c_L = \gamma$.

By playing the game by hand on $Q_n$ for small $n$, as well as running a cop-number-computation algorithm in `SageMath`, we have gained some modest insight into both $c$ and $c_L$. For instance, we know that

$$c(Q_5) = c_L(Q_5) = 2,$$
$$c(Q_6) = 2 < c_L(Q_6) = 3.$$

We also have a proof that $n \ge 19$ implies $c(Q_n) = 4$. We believe there to be a cutoff somewhere between $n = 6$ and $n = 19$ where $c(Q_n) = 4$ first happens, but we do not know precisely what it is. Proofs of this and related results will appear in a forthcoming paper. Meanwhile, here are some related conjectures and open problems.

- We know that $R_3$ is the unique smallest graph with $c_L = 3$, but the methods used to prove this are rather specific to $n = 3$ [**26**]. By asymptotic results [**24**], for large $n$ we know the hypercube graphs achieve $c_L = n$ with a smaller number of vertices than the rook's graphs $R_n$. In general, there is much work to be done to determine the smallest graph with $c_L = n$ (and $c = n$, for that matter).
- What is the smallest $n$ such that $c(Q_n) = 4$?
- We believe $c(Q_n) \le c(Q_{n+1})$ (and similarly for $c_L$), but a proof has eluded us. Of note, it would also make sense that $\gamma(Q_n) \le \gamma(Q_{n+1})$, but no one has proven that, either. Perhaps problems about queen's graphs are genuinely difficult!
- Which $n$ satisfy $c_L(Q_n) = \gamma(Q_n)$? Is there an $N$ such that this holds for all $n \ge N$?
- We know of no results for knight's graphs. Does a similar phenomenon occur where $c$ is bounded but $c_L$ grows with $n$? Is domination required for lazy cops to win?

We find it interesting that $c = 2$ yet $c_L = \gamma \gg c$ for both bishop's and rook's graphs. Is there is a property common to these two graph classes that can explain this? More generally, we are curious about the properties of graphs that make $c < c_L$. These particular chess graphs are the most striking examples we have found, as they make the difference as large as possible. This is certainly a broad topic; we welcome any relevant suggestions or results.

We would like to point out again that all of this research stems from a summer project conducted by the authors: a faculty mentor and two undergraduates. From our experience, games can be used to introduce enthusiastic mathematics majors to scholarly research and provide them with a rewarding intellectual pursuit [**7, 20**]. There is not necessarily much background knowledge required for students to get involved. We hope that instructors reading this will consider using Cops and Robbers and related topics in their courses and that students reading this may be motivated to try their hand at some of the open problems mentioned here.

**Acknowledgment.** The authors wish to thank Deepak Bal for inspiration at the outset of this project, as well as Stewart Neufeld for mailing us a copy of his master's thesis [**21**]. We also thank Emmanuel College for their generous funding to pursue this project.

**Summary.** Cops and Robbers is a classic pursuit–evasion game played on graphs. A new variant, Lazy Cops and Robbers, allows only one cop to move at a time, making the game's mechanics more akin to chess. We investigate and describe a few examples, showing that on some graphs lazy cops can be as effective as ordinary cops, but on other graphs they are not (and sometimes they are much worse). Some examples are based on the cops and robber moving like particular chess pieces. We also mention related results and pose some open problems.

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